Quadratic Equations

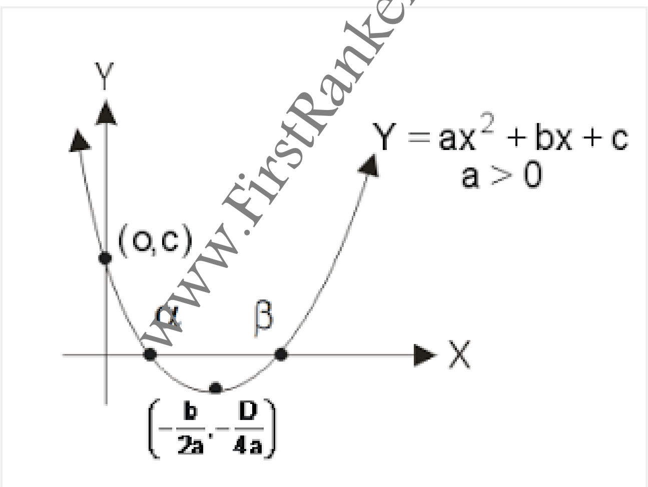
1. An expression of the form $ax^2 + bx + c$ where a, b, c are complex numbers and $a \neq 0$ is called a quadratic expression.

2. An equation of the form $ax^2 + bx + c = 0$ where a, b,c are complex numbers and $a \neq 0$ is called a quadratic equation.

3. A number α is called a root of the quadratic equation $ax^2 + bx + c = 0$ iff $a\alpha^2 + b\alpha + c = 0$

Roots of the Quadratic Equation:

4. The roots of the quadratic equation are $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.



The point $\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$ is the minima of this graph. i.e., at $-\dfrac{b}{2a}$ the graph takes it minimum value $-\dfrac{D}{4a}$

5. The quadratic equation $ax^2 + bx + c = 0$ have only the above two roots. If $b^2 - 4ac \neq 0$, then the two roots are different. If then the two roots are equal and the root is called a double root.

6. If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then $\alpha+\beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$, and $ax^2 + bx + c = a(x-\alpha)(x-\beta)$

7. Newton's Theorem: (Very useful Theorem)

If α, β are roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then for n > 2, n ∈N, we have
$aS_n + bS_{n-1} + cS_{n-2} = 0$

Example: If α, β are the roots of x^2 - 2x + 4 = 0, then the value of $\alpha^5 + \beta^5$ is

We know that $S_1 = \alpha + \beta = 2$ and $S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2.\alpha.\beta$ = 4 - 2(4) = - 4

Also a = 1; b = -2; c = 4

From Newton's Theorem

$aS_n + bS_{n-1} + cS_{n-2} = 0$

1 .S_3 + (-2).(S_2) + 4.S_1 $\Rightarrow S_3 - 2(-4) + 4(2) = 0$ $\Rightarrow S_3$ = - 16

1 .S_4 + (-2).(S_3) + 4.S_2 $\Rightarrow S_4 - 2(-16) + 4(-4) = 0$ $\Rightarrow S_4$ = - 16

1 .S_5 + (-2).(S_4) + 4.S_3 $\Rightarrow S_5 - 2(-16) + 4(-16) = 0$ $\Rightarrow S_5$ = 32

Condition to have same roots for two quadratic equations:

8. The quadratic equation $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ have the same roots if a : b: c =. $a_1 : b_1 : c_1$

Roots of the Quadratic Equation:

9. The quadratic equation whose roots are α, β is $(x - \alpha)(x - \beta) = 0$ i.e., $x^2 - (\alpha + \beta)x + \alpha\beta = 0.$

Example:

If $2x^2$ - 10x + 5 = 0, then the roots are

Here a = 2; b = -10; c = 5

The roots are = $\dfrac{-(-10) \pm \sqrt{(-10)^2 - 4.2.5}}{2.2} = \dfrac{10 \pm 2\sqrt{15}}{4} = \dfrac{5 \pm \sqrt{15}}{2}$

Discriminant of the Quadratic Equation:

10. The expression $b^2 - 4ac$ is called the descriminant of the quadratic equation $ax^2 + bx + c$ = 0 and it is denoted by Δ.

Example:

The discriminant of the equation x^2 + 2x - 3 = 0 is

Here a = 1, b = 2 and c = -3

The discriminant of the equation x^2 + 2x - 3 = 0 is = b^2 - 4ac = 4 - 4(1) (-3) = 16

Graphs of the Quadratic Equation:

11. If in the quadratic equation $ax^2 + bx + c = 0$, the numbers a,b,c are rational and a > 0 then the nature of the roots is as follows.

i. If $b^2 - 4ac > 0$, and $b^2 - 4ac$ is a perfect square i.e., $\sqrt{b^2 - 4ac}$ is a rational number, then the roots are rational and not equal. (Fig 1)

ii. If $b^2 - 4ac > 0$, and $b^2 - 4ac$ is not a perfect square i.e., $\sqrt{b^2 - 4ac}$ is a surd, then the roots are real and not equal. In fact the roots are two conjugate surds. (Fig 1)

iii. If $b^2 - 4ac = 0$ then the roots are rational and equal. (Fig -2)

iv. If $b^2 - 4ac < 0$, then the $\sqrt{b^2 - 4ac}$ is imaginary. The roots are complex numbers (imaginary) and not equal. In fact the roots are two conjugate complex numbers. (Fig - 3)

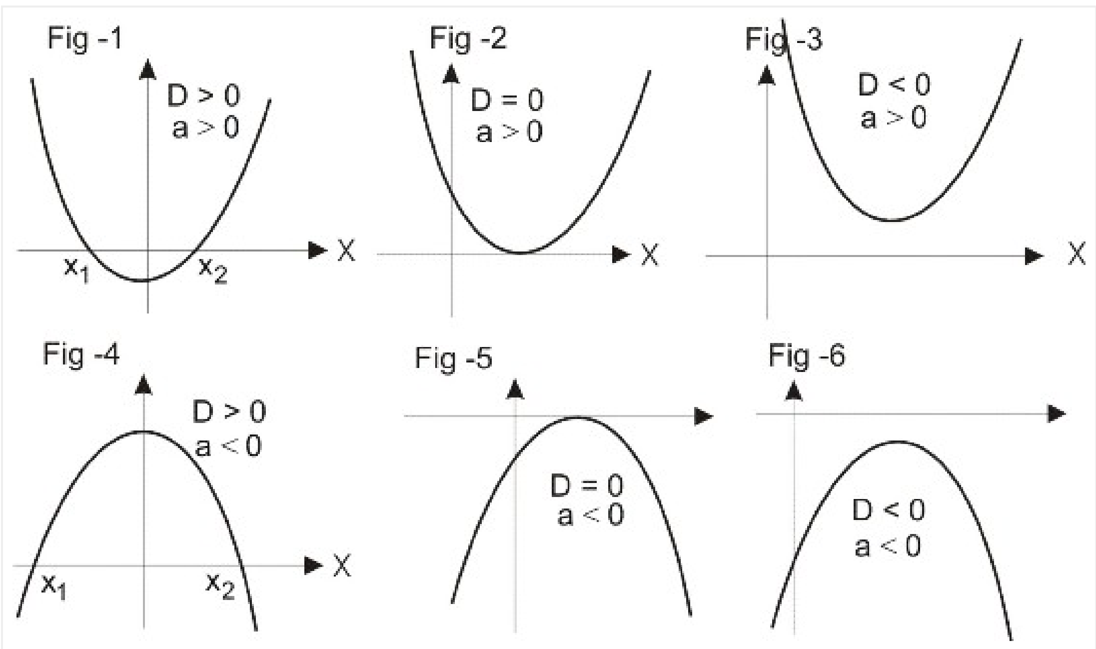
12. If in the quadratic equation $ax^2 + bx + c = 0$, the numbers a,b,c are rational and a < 0 then the nature of the roots is as follows.

i. If $b^2 - 4ac > 0$, and $b^2 - 4ac$ is a perfect square i.e., $\sqrt{b^2 - 4ac}$ is a rational number, then the roots are rational and not equal. (Fig -4)

ii. If $b^2 - 4ac > 0$, and $b^2 - 4ac$ is not a perfect square i.e., $\sqrt{b^2 - 4ac}$ is a surd, then the roots are real and not equal. In fact the roots are two conjugate surds. (Fig -4)

iii. If $b^2 - 4ac = 0$ then the roots are rational and equal. (Fig -5)

iv. If $b^2 - 4ac < 0$, then the $\sqrt{b^2 - 4ac}$ is imaginary. The roots are complex numbers (imaginary) and not equal. In fact the roots are two conjugate complex numbers. (Fig - 6)



13. In the quadratic equation $ax^2 + bx + c = 0$, $(a, b, c \in R)$

i. If a and c have the same sign then both the roots are positive or both the roots are negative.

ii. If a and c have opposite signs then the roots have opposite signs.

iii. If a + b+ c = 0, then the roots are 2, $\frac{c}{a}$

iv. If a - b + c = 0, then the roots are $-1,\ -.\frac{c}{a}$

v. If the two roots are negative then a,b,c have the same sign.

vi. If the two roots are positive then the sign of a,c is different from the sign of b.

14. The necessary and sufficient condition that the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$.

15. Let be the roots of the quadratic equation $ax^2 + bx + c$ = 0.

i. If α, β are imaginary $(\Delta = 0)$ then for all real values of x :$ax^2 + bx + c$, a have the same sign.

ii. If α, β , are real and equal $(\Delta = 0)$ then for all real values of x except α: $ax^2 + bx + c$, a have the same sign.

iii. If α, β , are real and not equal $(\Delta > 0)$ such that $\alpha < \beta$ then

a. $\alpha < x < \beta \Rightarrow ax^2 + bx + x,$ a have opposite signs

b. $x < \alpha or \beta < x \Rightarrow ax^2 + bx + c,$, a have the same sign

16. Let f(x) = $ax^2 + bx + c$ be a quadratic function. Then

i. If a > 0, then f(x) has minimum value at $x = \frac{-b}{2a}$ and minimum value = $\frac{4ac - b^2}{4a}$.

ii. If a < 0, then f(x) has minimum value at $x = \frac{-b}{2a}$ and maximum value = $\frac{4ac - b^2}{4a}$.

Solved Examples

If are the roots of x^2 - 2x - 1 = 0, then the value of $\alpha^2 + \beta^2$ is

The Q.E is x^2 - 2x - 1 = 0

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$ = 2,

Product of the roots = $\alpha.\beta$ = $\frac{c}{a}$ = -1

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2.\alpha.\beta$ = 4 + 2 = 6

The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots . \infty}}}$ is

Let x = $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots\ldots\ldots . \infty}}}$

Since the terms go on infinity, the given quantity will not change if we omit the radical before $\sqrt{6}$, and those after the first one are taken to be equal to x.

Hence we have x = $\sqrt{6 + x}$

Squaring both sides, we get

$x^2 = 6 + x \Rightarrow x^2 - x - 6 = 0$

$\Rightarrow$ (x- 3) (x + 2) = 0,x = 3 or -2

but the given quantity is positive. Hence x = 3

In copying a quadratic equation of the form $x^2 + px + q = 0$, the coefficient of x was wrongly written as -10 instead of -11 and the roots were found to be 4 and 6. The roots of the correct equation are ....

$x^2 + px + q = 0$

$x^2 - (10x) + 24 = 0$

Comparing with $x^2 - (a + b)x + ab = 0$,

It is wrongly written as -10 instead of -11

So in actual - (a + b) = 11

(a + b) = -(8 + 3). So the roots are 8, 3.

If α and β are the roots of the equation (x - a)(x - b) + c = 0, find the roots of the equation $(x - \alpha)(x - \beta) = c$

(x - a)(x - b) + c = 0

$x^2 - bx - ax + ab + c = 0$

$x^2 - (b + a)x + (ab + c) = 0$ ...(1)

Comparing with $x^2 - (\alpha + \beta)x + \alpha.\beta = 0$

$\alpha + \beta = a + b$

$\alpha.\beta = ab + c$

Similarly,

$(x - \alpha)(x - \beta) = c$
$x^2 - \beta x - \alpha x + \alpha\beta = c$
$x^2 - (\alpha + \beta)x + \alpha\beta = c$
$x^2 - (a + b)x + ab = 0$

So, the roots are a, b.

The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite sign is

$3x^2 + 2x + p(p - 1) = 0$

$x^2 + (2/3)x + [p(p - 1)/3] = 0$

as the roots are of opposite sign,

$\alpha.\beta < 0$

$[p(p - 1)/3] < 0$

$p(p - 1) < 0$

i.e., $0 < p < 1$.